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Spectral Properties of Products of Projections in Quantum Probability Theory

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Starting from the fact that, for projections P and Q in Hilbert space, equality of PQ and QP (i.e., commutativity) is equivalent to the equality of the triple products PQP and QPQ, the spectral resolutions of PQP and QPQ for not necessarily commuting projections are compared. It is shown that the respective eigenspaces are isometric and display a curious biorthogonality to be described below. A more general setup relates spectral properties of operators TT^* and T^*T for bounded T. The result is connected with Mittelstaedt's theory of a probability theory for quantum mechanics.

1. INTRODUCTION

In his stimulating book *Philosophische Probleme der modernen Physik*, Professor Mittelstaedt (1976) sketches an axiomatic probability theory based on his quantum logics. He shows that the function w_φ, defined for a fixed state vector φ by

$$w_\varphi(A)=\langle\varphi,P_A\varphi\rangle$$

fulfills his axioms. Here, and in the following, P_A is the projection onto a closed linear subspace of the underlying (complex) Hilbert space H whose inner product is denoted by $\langle\ ,\ \rangle$.

In the course of an objectivistic interpretation of Young's interference experiment, Mittelstaedt introduces the operator $P_BP_AP_B$ for noncommuting projections P_A and P_B. This self-adjoint operator defines a probability

$$w_\varphi(A,B)=\langle\varphi,P_BP_AP_B\varphi\rangle$$

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which, by way of the formula

$$w_\varphi^{\text{int}}(A,B)=w_\varphi(A)-w_\varphi(A,B)-w_\varphi(A,\urcorner B)$$

determines the probability of interference $w_\varphi^{\text{int}}(A,B)$. The aim of this paper will be the study of operators of type $P_B P_A P_B$. In particular, we seek to compare this operator with its counterpart $P_A P_B P_A$, and we will see that, if these last two triple products coincide, P_A and P_B commute.

We shall show that even for noncommuting projections P_A and P_B, the spectral resolutions of $P_A P_B P_A$ and $P_B P_A P_B$ lead to isometric eigenspaces for all nonzero spectral values. As a consequence, the discrete and the continuous spectra of $P_A P_B P_A$ and $P_B P_A P_B$ are identical. The difference between these two observables lies solely in their probabilistic structures, which, however, are closely interrelated and which determine each other completely.

2. A LEMMA ON COMMUTATIVITY OF P_A AND P_B

In Mittelstaedt's probability theory, the probability of interference vanishes for commuting projections P_A and P_B. Since the triple product $P_B P_A P_B$ enters the defining formula for the interference term, it is suggestive that $P_B P_A P_B = P_A P_B P_A$ should already imply commutativity. This is indeed true:

Lemma 2.1. For any two projections P,Q in H, the relation $PQ=QP$ is equivalent to $PQP=QPQ$.

Our proof depends upon the following lemma.

Lemma 2.2. Let A and B be linear bounded operators in H such that

(1) $AB=BA$
(2) $A^2=B^2$
(3) $(A-B)=-(A-B)^*$

Then A and B are related by $A=(2E-I)B$, where E is the orthogonal projection onto the null space $M=N(A-B)$.

Proof of Lemma 2.2 cf. Bachman and Narici, 1966, p. 424, Theorem 23.3; note, however, that our proof does not require self-adjointness of A

and B; instead, we need (3). From (1) and (2) we have

$$(A-B)(A+B)=A^2-B^2=0$$

i.e.

$$(4)\quad E(A+B)=A+B$$

For any vector $z \in H$ write $z=x+y$, where $x \in M$ and $y \in M^{\perp}$. It follows

$$E(A-B)z=E(A-B)x+E(A-B)y$$

The first term on the right is zero, because $x \in M=N(A-B)$, and the second is equal to $(A-B)Ey$, using assumption (3) and the same commutativity argument as in Bachmann and Narici, p. 425. Therefore

$$(5)\quad E(A-B)=0$$

Combining (4) and (5), we have

$$E(A+B)-E(A-B)=A+B$$

or

$$A=2EB-B=(2E-I)B \qquad \blacksquare$$

We wish to apply Lemma 2.2 for $A=PQ$, $B=QP$. Assumption (1) is the same as $PQP=QPQ$. Using this, and observing

$$(PQ)^2=PQPQ=PPQP=PQP=QPQ=(QP)^2$$

we note that (2) is fulfilled. Moreover $(PQ-QP)^*=QP-PQ=-(PQ-QP)$, which is assumption (3).

Proof of Lemma 2.1. Obviously, the first equality implies the second. Assume $PQP=QPQ$. Applying Lemma 2.1, which is "symmetrical" in PQ and QP, gives

$$PQ=(2E-I)QP \quad \text{and} \quad QP=(2E-I)PQ$$

where E is the projection onto

$$N(PQ-QP)=N(QP-PQ)$$

We prove now $PQ=QP$.

Write M_P and M_Q for the ranges of P and Q, and denote by $\vee$ and $\wedge$ the closed span and the intersection of (closed) subspaces in H.

If $z\in M_P\wedge M_Q$, trivially $PQz-QPz=z-z=0$. If $z\in M_P^{\perp}\vee M_Q^{\perp}$, write $z=x+y, x\in M_P^{\perp}, y\in M_Q^{\perp}$. Consequently,

$$PQz=PQx+PQy=PQx+0=(2E-I)QPx+0=0$$

Similarly,

$$QPz=QPx+QPy=0+QPy=0+(2E-I)PQy=0$$

If z is the limit of x_n+y_n, also $PQz=QPz=0$ from the continuity of PQ and QP. Hence, PQ and QP coincide on all of H. ■

If we once more write $A=PQ$ and for the conjugate $A^*=QP$, then Lemma 2.1 expresses the fact that $A=A^*$ and $AA^*=A^*A$ are equivalent: normality of $A=PQ$ and self-adjointness are the same.

The following pages are meant to show how the difference between PQ and QP (noncommutativity) is expressed in differences occurring in the spectral resolutions of the self-adjoint operators PQP and QPQ.

3. THE SPECTRAL RESOLUTIONS OF TT^* AND T^*T

In this section, T denotes a bounded operator in H. Let S be another bounded operator in H, and write $\sigma(ST)$ and $\sigma(TS)$ for the spectrum of ST and TS. Assume throughout that 0 is a spectral value (cf. Remark 3.1).

Lemma 3.1.

$$\sigma(ST)=\sigma(TS)$$

Proof. This follows from the fact that, if $I-TS$ is invertible, the inverse of $I-ST$ is given by $I+S(I-TS)^{-1}T$ (cf. Hirzebruch and Scharlau, 1971, p. 120). ■

Corollary 3.1. $\sigma(T^*T)=\sigma(TT^*)$; in particular for $T=PQ$: $\sigma(QPQ)=\sigma(PQP)$

Let us write $R(T)$ and $N(T)$ for the closed range and the null space of an operator T. Since $TT^* \geqslant 0$ is self-adjoint, it has a spectral resolution E_{TT^*} such that

$$TT^* = \int_{\sigma(TT^*)} \lambda E_{TT^*}(d\lambda)$$

with $\sigma(TT^*) = \sigma(T^*T) \subset [0, \infty)$. Our aim is to relate E_{TT^*} to the spectral resolution E_{T^*T} of T^*T. First, we shall determine the ranges of $E_{TT^*}(0)$ and $E_{T^*T}(0)$.

Lemma 3.2.

$$(1)\ R(E_{TT^*}(0)) = R(T)^{\perp} = N(T^*)$$

$$(2)\ R(E_{T^*T}(0)) = R(T^*)^{\perp} = N(T)$$

Proof. (1) It is well known that $R(E_{TT^*}(0)) = N(TT^*)$. From $TT^*x = 0$ we get $\langle TT^*x, x\rangle = \|T^*x\|^2 = 0$, and therefore $x \in N(T^*)$, which is equal to $R(T)^{\perp}$. The converse inclusion is obvious. (2) substitute T^* for T in (1). ∎

Let us specialize Lemma 3.2 for $T = PQ$. It is clear that $\sigma(PQP) \subset [0, 1]$.

Lemma 3.3.

$$(1')\ R(E_{PQP}(0)) = R(P)^{\perp} \vee [R(P) \wedge R(Q)^{\perp}]$$

$$(2')\ R(E_{QPQ}(0)) = R(Q)^{\perp} \vee [R(Q) \wedge R(P)^{\perp}]$$

($\vee$ denotes the closed span of two subspaces in H, and $\wedge$ their intersection.)

Proof. (1′) $T^*x = QPx = 0$ is equivalent to $Px \in R(Q)^{\perp}$, and this is certainly fulfilled for all x in the subspace on the right-hand side of (1′). Conversely, if x is such that $Px \in R(Q)^{\perp}$, then we see from $x = (I-P)x + Px$ and $R(I-P) = R(P)^{\perp}$ that x belongs to the right-hand side of (1′). (2′) is proved the same way. ∎

Remark 3.1. For projections P, Q not equal to the identity operator I the point 0 is always a spectral value: otherwise the null space of PQP, which is identical with $R(E_{PQP}(0))$, contains only the 0-vector. This, however, would mean $R(P)^{\perp} = R(Q)^{\perp} = \{0\}$, or $P = Q = I$. To avoid the trivial case $P = Q = I$ we may therefore assume 0 to be in $\sigma(PQP) = \sigma(QPQ)$.

We shall pursue this special case $T=PQ$ further by evaluating the ranges $E_{PQP}(1)$ and $E_{QPQ}(1)$.

Lemma 3.4. $R(E_{PQP}(1))=R(P)\wedge R(Q)=R(E_{QPQ}(1))$.

Proof. $x\in R(E_{PQP}(1))$ is characterized by $PQPx=x$. From $\|x\|^2=\langle PQPx,x\rangle=\|QPx\|^2\leqslant\|Px\|^2\leqslant\|x\|^2$ we see that $\|Px\|=\|x\|$ and therefore $Px=x$, i.e., $x\in R(P)$. It follows also that $QPx=Px$, and thus together with $Px=x$ that $Qx=x$, i.e., $x\in R(Q)$. The converse is evident, and the second equality follows by symmetry. ∎

Remark 3.2. The number 1 is not a spectral value if and only if $R(P)\wedge R(Q)=\{0\}$. This happens in particular if (but not only if) $PQ=QP=0$, i.e., $R(P)\perp R(Q)$.

Interpretation. Lemmas 3.3 and 3.4 can be interpreted in terms of Mittelstaedt's "Quantenlogik." Lemma 3.3 (2′), e.g., tells us that $R(E_{QPQ}(0))$ with $P=P_A$ and $Q=P_B$ corresponds to the implication $B\rightarrow\neg A$ (cf. Mittelstaedt, 1976, p. 204), and so the support of E_{QPQ}, i.e., the orthocomplement of $R(E_{QPQ}(0))$ represents $\neg(B\rightarrow\neg A)$. Hence we can interpret

$$w_\varphi(A,B)=\int\lambda\langle E_{QPQ}(d\lambda)\varphi,\varphi\rangle$$

as the expected value that "in the state φ it is not true that B implies not—A." Lemma 3.4 means that, in case 1 is an eigenvalue, the corresponding eigenspaces of $\lambda=1$ are identical for PQP and QPQ and equal to $R(P)\wedge R(Q)$. $\langle E_{PQP}(1)\varphi,\varphi\rangle$ is the probability that $A\wedge B$ occurs. The inclusion $R(E_{QPQ}(1))\subset R(E_{QPQ}(0))^\perp=R(E_{QPQ}(0,1])$ has its exact quantum-logic counterpart in $A\wedge B\leqslant\neg(B\rightarrow\neg A)=B\wedge(\neg B\vee A)$.

4. AN ISOMETRY BETWEEN $R(E_{TT^*}(X))$ AND $R(E_{T^*T}(X))$ FOR $X\subset(0,\infty)$

Let T again be bounded, and $X\subset(0,\infty)$ be a Borel set. In this section we shall establish a partial isometry between the ranges of $E_{TT^*}(X)$ and $E_{T^*T}(X)$.

In fact, the existence of a partial isometry follows from generalities known in connection with the square root of positive operators (cf. for the following Rudin, 1973, pp. 313–316). T^*T is positive, and its self-adjoint square root is the unique operator $(T^*T)^{1/2}$ which obeys the norm relation

$$\|Tx\| = \|(T^*T)^{1/2}x\|, \qquad x \in H$$

Consequently the null spaces of T and $(T^*T)^{1/2}$ coincide:

$$N(T) = N((T^*T)^{1/2})$$

and the range of $(T^*T)^{1/2}$ equals $R(T^*) = N(T)^{\perp}$ (cf. Lemma 3.2 and its proof), so that

$$R(T^*) = R((T^*T)^{1/2}) = R(E_{T^*T}(0))^{\perp} = R(E_{T^*T}(0,\infty)) \tag{4.1}$$

The formula

$$V_{T^*T}(T^*T)^{1/2}x = Tx \tag{4.2}$$

defines an isometry V_{T^*T} from $R(E_{T^*T}(0))^{\perp}$ onto $R(T)$, which, by Lemma 3.2 (1), equals $R(E_{TT^*}(0))^{\perp}$. V_{T^*T} may be extended to a bounded operator on H by defining $V_{T^*T}y = 0$ for all $y \in R(E_{T^*T}(0))$; thus V_{T^*T} becomes a partial isometry on H. The same reasoning applies to the square root $(TT^*)^{1/2}$ of TT^*, and via

$$V_{TT^*}(TT^*)^{1/2}x = T^*x \tag{4.3}$$

we obtain a partial isometry V_{TT^*} from $R(T) = R(E_{TT^*}(0))^{\perp}$ onto $R(T^*) = R(E_{T^*T}(0))^{\perp}$, which vanishes on $R(T)^{\perp} = N(T^*) = R(E_{TT^*}(0))$. By definition

$$V_{T^*T} = V_{TT^*}^* \tag{4.4}$$

On the support of $(TT^*)^{1/2}$, i.e., on $R(E_{TT^*}(0))^{\perp}$, we can write (4.4) in the form

$$V_{TT^*} = T^*(TT^*)^{-1/2} \tag{4.5}$$

or, using the spectral representation for $(TT^*)^{-1/2}$:

$$V_{TT^*} = T^* \int_{\lambda>0} \lambda^{-1/2} E_{TT^*}(d\lambda) \tag{4.6}$$

On the other hand, using (4.4), $V_{TT^*} = V_{T^*T}^*$ admits the representation

$$V_{TT^*} = (T^*T)^{-1/2}T^* \tag{4.7}$$

or, with the spectral representation for $(T^*T)^{-1/2}$:

$$V_{TT^*} = \int_{\lambda>0} \lambda^{-1/2} E_{T^*T}(d\lambda)\, T^* \tag{4.8}$$

(4.5) and (4.7) express a curious commutation rule which will play a role later on.

The respective counterparts for V_{T^*T} are given by

$$V_{T^*T} = T(T^*T)^{-1/2} \tag{4.9}$$

$$V_{T^*T} = T \int_{\lambda>0} \lambda^{-1/2} E_{T^*T}(d\lambda) \tag{4.10}$$

$$V_{TT} = (TT^*)^{-1/2}\, T \tag{4.11}$$

$$V_{T^*T} = \int_{\lambda>0} \lambda^{-1/2} E_{TT^*}(d\lambda)\, T \tag{4.12}$$

As a first result, then, there is an isometry between the ranges of $E_{TT^*}(X)$ and $E_{T^*T}(X)$ for the special Borel set $X = (0, \infty)$.

For the general case, define for any Borel set $X \subset (0, \infty)$

$$V_{TT^*}(X) := V_{TT^*} E_{TT^*}(X) \tag{4.13}$$

$$V_{T^*T}(X) := V_{T^*T} E_{T^*T}(X) \tag{4.14}$$

From the definition of V_{TT^*} and V_{T^*T} it is clear that $V_{TT^*}(X)$ and $V_{T^*T}(X)$ are partial isometries with domains $R(E_{TT^*}(X))$ and $R(E_{T^*T}(X))$, respectively. Furthermore, to determine the ranges of $V_{TT^*}(X)$ and $V_{T^*T}(X)$, note that the adjoint of $V_{TT^*}(X)$ [of $V_{T^*T}(X)$] is a partial isometry whose domain is equal to the range of $V_{TT^*}(X)$ [of $V_{T^*T}(X)$].

The adjoint of $V_{TT^*}(X)$, however, is $V_{T^*T}(X)$! In order to prove this, we need a simple lemma.

Lemma 4.1.

(1) $E_{TT^*}(X)\, T = T E_{T^*T}(X)$

(2) $T^* E_{TT^*}(X) = E_{T^*T}(X)\, T^*$

Proof. This is a special case of Fuglede's theorem; cf. Radjavi and Rosenthal, p. 20. ■

Now we are ready to prove the following.

Lemma 4.3. $V^*_{TT^*}(X) = V_{T^*T}(X)$.

Proof.

$$\begin{aligned} V^*_{TT^*}(X) &= E_{TT^*}(X)\,V^*_{TT^*} = E_{TT^*}(X)\,V_{T^*T} \\ &= E_{TT^*}(X)T(T^*T)^{-1/2} \\ &= TE_{T^*T}(X)(T^*T)^{-1/2} \\ &= T(T^*T)^{-1/2}E_{T^*T}(X) = V_{T^*T}E_{T^*T}(X) \\ &= V_{T^*T}(X) \end{aligned} \qquad \blacksquare$$

Here we have used Lemma 4.1 and the fact that $E_{T^*T}(X)$ commutes with $(T^*T)^{-1/2}$.

We have found the following theorem.

Theorem 4.1. For every Borel set $X \subset (0, \infty)$ the operators $V_{TT^*}(X)$ and $V_{T^*T}(X)$ define isometries

$$R(E_{TT^*}(X)) \underset{V_{T^*T}(X)}{\overset{V_{TT^*}(X)}{\leftrightarrows}} R(E_{T^*T}(X))$$

with $V^*_{TT^*}(X) = V_{T^*T}(X)$.

Specializing for $T = PQ$, we obtain from Lemma 4.1 the following corollary.

Corollary 4.1.

$$(1')\ E_{PQP}(X)PQ = PQE_{QPQ}(X)$$

$$(2')\ QPE_{PQP}(X) = E_{QPQ}(X)QP$$

These two equalities may be simplified by observing that $E_{PQP}(X) \leqslant P$, $E_{QPQ} \leqslant Q$ or, equivalently, that $E_{PQP}(X)P =$

$PE_{PQP}(X) = E_{PQP}(X)$ and $E_{QPQ}(X)Q = QE_{QPQ}(X) = E_{QPQ}(X)$. This yields immediately the following additional corollary.

Corollary 4.2.

$$(1'')\ E_{PQP}(X)Q = PE_{QPQ}(X)$$

$$(2'')\ QE_{PQP}(X) = E_{QPQ}(X)P$$

These last two relations provide further insight into the relationship between the spectral resolutions of PQP and QPQ: it is known from general properties of spectral measures that for disjoint Borel sets X, Y

$$E_{PQP}(X) \perp E_{PQP}(Y) \tag{4.15}$$

$$E_{QPQ}(X) \perp E_{QPQ}(Y) \tag{4.16}$$

It is remarkable that these relations remain true if the projections on the right-hand side of (4.15) and (4.16) are interchanged.

Corollary 4.3. For disjoint Borel sets X, $Y \subset (0, \infty)$

$$E_{PQP}(X) \perp E_{QPQ}(Y) \tag{4.17}$$

$$E_{QPQ}(X) \perp E_{PQP}(Y) \tag{4.18}$$

Proof. Multiply (2″) of Corollary 4.2 by $E_{QPQ}(Y)$ from the left, so that by (4.16) the right-hand side of (2″) vanishes: $E_{QPQ}(Y)QE_{PQP}(X) = E_{QPQ}(Y)E_{PQP}(X) = 0$, which is (4.17); (4.18) follows similarly. ∎

Remark 4.1. The main result of Section 4, Theorem 4.1, proves that TT^* and T^*T not only have identical spectra (Corollary 3.1), but that their discrete and continuous spectra coincide. Furthermore, for a Borel set $X \subset [0, \infty)$ the ranges of the projections $E_{TT^*}(X)$ and $E_{T^*T}(X)$ have the same dimension! If X does not contain 0, this follows from the partial isometry of their ranges (Theorem 4.1), and in particular, if $X = (0, \infty)$, for the support of TT^* and T^*T. From this, however, it follows also for the orthocomplements $R(E_{TT^*}(0))$ and $R(E_{T^*T}(0))$, that is, equidimensionality also for $X = \{0\}$.

5. THE CASE WHERE H HAS FINITE DIMENSION

In this section we intend to discuss the general results obtained so far for the special case where H is a finite-dimensional Hilbert space.

To begin with, consider the case that P and Q are projections with one-dimensional ranges $R(P)$ and $R(Q)$. If x_0 and y_0 are generating unit vectors in $R(P)$ and $R(Q)$, then, for all $x \in H$

$$Px = \langle x, x_0 \rangle x_0$$

$$QPx = \langle x, x_0 \rangle \langle x_0, y_0 \rangle y_0$$

$$PQPx = \langle x, x_0 \rangle \langle x_0, y_0 \rangle \langle y_0, x_0 \rangle x_0$$

or

$$PQP = |\langle x_0, y_0 \rangle|^2 P \tag{5.1}$$

Similarly

$$QPQ = |\langle x_0, y_0 \rangle|^2 Q \tag{5.2}$$

From this we obtain a strengthening of Lemma 2.1.

Lemma 5.1. For one-dimensional projections P, Q with $PQ \neq 0$

$$P = Q \Leftrightarrow PQ = QP \Leftrightarrow PQP = QPQ$$

(This result does not depend on finite dimensionality of H.) From now on, H is assumed to be finite dimensional, and the ranges $R(P)$ of P and $R(Q)$ of Q may have different dimensions. As a consequence of finite dimensionality, the spectrum $\sigma(PQP) = \sigma(QPQ)$ consists of finitely many eigenvalues λ_i, $i = 1, 2, \ldots, k$, only, and the respective spectral resolutions of PQP and QPQ may be written as

$$PQP = \sum_{i=1}^{k} \lambda_i E_{PQP}(\lambda_i)$$

$$QPQ = \sum_{i=1}^{k} \lambda_i E_{QPQ}(\lambda_i)$$

where by virtue of Theorem 4.1, the ranges of $E_{PQP}(\lambda_i)$ and $E_{QPQ}(\lambda_i)$, $i = 2, 3, \ldots, k$ are isometric, and from Remark 4.1 we also know that the null spaces $N(PQP) = R(E_{PQP}(0))$ and $N(QPQ) = R(E_{QPQ}(0))$ are isomorphic.

Note also that in this case the lattice of subspaces of H is modular (cf. Jauch, 1973, p. 84). This means in particular that for $R(P)^\perp \subset R(Q)^\perp$ or, equivalently, for $R(Q) \subset R(P)$

$$N(PQP) = R(Q)^\perp$$

On the other hand, since $R(Q)\subset R(P)$ implies $R(Q)\wedge R(P)^{\perp}=\{0\}$, we also have

$$N(QPQ)=R(Q)^{\perp}$$

As a consequence, $R(E_{PQP}(0))^{\perp}$ and $R(E_{QPQ}(0))^{\perp}$ are also identical and equal to $R(Q)$, so that the isometry from Theorem 4.1 is simply the identity map, and our elaborate correspondence as established in Section 4 collapses into a triviality. This is of course to be expected from the fact that $R(Q)\subset R(P)$ means $QP=Q=QP$, i.e., Q and P commute, in which case we are not interested.

We now make the further assumption that, with the possible exception of the null spaces $N(PQP)$ and $N(QPQ)$, the ranges of all projections $E_{PQP}(\lambda_i)$ and $E_{QPQ}(\lambda_i)$, $i=2,3,\ldots,k$ are one dimensional. Let $x_i\in R(E_{PQP}(\lambda_i))$ and $y_i\in R(E_{QPQ}(\lambda_i))$ be unit vectors, $\|x_i\|=\|y_i\|=1$, $i=2,3,\ldots,k$. $\{x_i\}$ and $\{y_i\}$ thus constitute a basis for the spaces $R(E_{PQP}(0))^{\perp}=R(E_{PQP}(\lambda_2))+\cdots+R(E_{PQP}(\lambda_k))$ and $R(E_{QPQ}(0))^{\perp}=R(E_{QPQ}(\lambda_2))+\cdots+R(E_{QPQ}(\lambda_k))$, and we have, by Corollary 4.3,

$$x_i\perp x_j,\qquad y_i\perp y_j,\qquad x_i\perp y_j,\qquad i\neq j,\qquad i,j=2,3,\ldots,k \tag{5.3}$$

These orthogonality properties have a consequence on the relationship between the probabilistic structures of PQP and QPQ. In order to derive these consequences, write

$$\varphi_1:=E_{PQP}(0)^{\perp}\varphi=\sum_{i=1}^{k}\langle\varphi,x_i\rangle x_i$$

$$\varphi_2:=E_{QPQ}(0)^{\perp}\varphi=\sum_{i=1}^{k}\langle\varphi,y_i\rangle y_i$$

this just means that we consider only that part of the state vector φ that belongs to the support of PQP and QPQ, respectively.

Using the relations (5.3) above yields

$$\left.\begin{aligned}\langle E_{PQP}(0)^{\perp}\varphi,y_i\rangle&=\langle\varphi,x_i\rangle\langle x_i,y_i\rangle \qquad (5.4)\\ \langle E_{QPQ}(0)^{\perp}\varphi,x_i\rangle&=\langle\varphi,y_i\rangle\langle y_i,x_i\rangle \qquad (5.5)\end{aligned}\right\}\quad i=2,3,\ldots,k$$

and from this

$$\left.\begin{aligned}|\langle\varphi,E_{PQP}(0)^{\perp}y_i\rangle|^2&=w_\varphi(x_i)|\langle x_i,y_i\rangle|^2 \qquad (5.6)\\ |\langle\varphi,E_{QPQ}(0)^{\perp}x_i\rangle|^2&=w_\varphi(y_i)|\langle x_i,y_i\rangle|^2 \qquad (5.7)\end{aligned}\right\}\quad i=2,3,\ldots,k$$

Interpretation. The left-hand side of (5.6) is the probability that "λ_i is observed from the observable QPQ (represented by y_i) relative to the support of PQP [represented by $E_{PQP}(0)^{\perp}$], when the state of the system is given by φ." Denote this probability by $w_\varphi(y_i|PQP)$. An analogous interpretation can be given to $w_\varphi(x_i|QPQ)$ in (5.7). The two equations, (5.6) and (5.7), relate the probabilities $w_\varphi(x_i)=\langle\varphi, E_{PQP}(\lambda_i)\varphi\rangle$ and $w_\varphi(y_i)=\langle\varphi, E_{QPQ}(\lambda_i)\varphi\rangle$ via $w_\varphi(y_i|PQP)$ and $w_\varphi(x_i|QPQ)$.

Loosely speaking, PQP and QPQ represent "the same" experimental evidence in the sense that they allow exactly the same measurements (identical spectra). Yet, these measurements occur with different probabilities; or, in other words, the two random variables associated with PQP and QPQ have the same realizations but different distributions $\{w_\varphi(x_i)\}$ and $\{w_\varphi(y_i)\}$. These distributions, however, determine each other through (5.6) and (5.7), e.g.,

$$w_\varphi(y_i)=\frac{w_\varphi(x_i|QPQ)}{w_\varphi(y_i|PQP)}\,w_\varphi(x_i), \qquad i=2,3,\ldots,k$$

(provided there are no zero denominators).

6. THE CASE WHERE H IS TWO DIMENSIONAL

If H is only two dimensional and when P and Q are one-dimensional projections (cf. Mittelstaedt, 1976, pp. 134–141, 208–218) $P\neq Q$, we only have two eigenvalues

$$\lambda_1=0, \qquad \lambda_2=|\langle x_0,y_0\rangle|^2$$

and

$$E_{PQP}(\lambda_2)=P \tag{6.1}$$

$$E_{QPQ}(\lambda_2)=Q \tag{6.2}$$

$$R(E_{PQP}(\lambda_2))=R(P) \tag{6.3}$$

$$R(E_{QPQ}(\lambda_2))=R(Q) \tag{6.4}$$

Since here obviously $R(P)\subset R(Q)^{\perp}$ and $R(Q)\subset R(P)^{\perp}$, we have for the null spaces, according to Lemma 3.3:

$$R(E_{PQP}(0))=R(P)^{\perp} \tag{6.5}$$

$$R(E_{QPQ}(0))=R(Q)^{\perp} \tag{6.6}$$

and these are the orthocomplements of the spaces in (6.3) and (6.4). The isometry V_{PQP}: $R(P) \to R(Q)$ is in our present special case given by [cf. (4.8) above]

$$\begin{aligned} V_{PQP} &= \frac{1}{|\langle x_0, y_0 \rangle|} E_{QPQ}(\lambda_2) \cdot QP \\ &= |\langle x_0, y_0 \rangle|^{-1} Q \cdot QP \qquad [\text{by } (6.2)] \\ &= |\langle x_0, y_0 \rangle|^{-1} QP \end{aligned}$$

and V_{QPQ}: $R(Q) \to R(P)$ has the form

$$\begin{aligned} V_{QPQ} &= \frac{1}{|\langle x_0, y_0 \rangle|} E_{PQP}(\lambda_2) PQ \\ &= |\langle x_0, y_0 \rangle|^{-1} P \cdot PQ \qquad [\text{by } (6.1)] \\ &= |\langle x_0, y_0 \rangle|^{-1} PQ \end{aligned}$$

so that indeed $V_{QP} = V^*_{PQ}$.

For the simple case at hand, we can calculate all probabilities mentioned in Section 1 (in the following A and B are again propositions represented by $P = P_A$ and $Q = P_B$, and the unit vector φ represents the "state" of the physical system):

$$w_\varphi(A) = \langle \varphi, P_A \rangle = |\langle \varphi, x_0 \rangle|^2 \tag{6.7}$$

$$\begin{aligned} w_\varphi(A, B) &= \langle \varphi, P_B P_A P_B \varphi \rangle = \lambda_2 \langle E_{QPQ}(\lambda_2) \varphi, \varphi \rangle \\ &= |\langle x_0, y_0 \rangle|^2 \langle P_B \varphi, \varphi \rangle \qquad [\text{from } (6.2)] \\ &= w_B(A) w_\varphi(B) \qquad (\text{cf. Mittelstaedt, 1976, p. 214}) \end{aligned} \tag{6.8}$$

$$\begin{aligned} w_\varphi(A, \neg B) &= \langle \varphi, (I - P_B) P_A (I - P_B) \varphi \rangle \\ &= w_\varphi(A) - \langle \varphi, P_A P_B \varphi \rangle - \langle \varphi, P_B P_A \varphi \rangle + w_\varphi(A, B) \end{aligned} \tag{6.9}$$

and from these three equations we get for the probability of interference

$$\begin{aligned} w_\varphi^{\text{int}}(A, B) &= \langle \varphi, P_A P_B \varphi \rangle + \langle \varphi, P_B P_A \varphi \rangle - 2 \langle \varphi, P_B P_A P_B \varphi \rangle \\ &= 2\,\mathrm{Re}(\langle y_0, \varphi \rangle \langle x_0, y_0 \rangle \langle \varphi, x_0 \rangle - w_\varphi(A, B)) \end{aligned} \tag{6.10}$$

(cf. also Mittelstaedt, 1976, p. 140). Let P_A and P_B, $P_A P_B \neq 0$ be given.

Then it is reasonable to ask, for which state φ the probability of interference $w_\varphi^{\text{int}}(A,B)$ attains its extreme values.

In order to compute these values, we restrict the following discussion to the case where H is a *real* two-dimensional Hilbert space. Then we see that, for φ in the acute angle between x_0 and y_0, $w_\varphi^{\text{int}}(A,B)=2\langle x_0,y_0\rangle (\langle y_0,\varphi\rangle\langle\varphi,x_0\rangle-\langle x_0,y_0\rangle\langle y_0,\varphi\rangle^2)\geqslant 0$ is minimal or maximal if $\cos\beta(\cos\alpha - \cos\gamma\cos\beta)$ is. (Here α is the angle between x_0 and φ, β is the angle between φ and y_0, and $\gamma=\alpha+\beta$.) Using the trigonometric identity

$$\cos\alpha=\cos(\gamma-\beta)=\cos\gamma\cos\beta+\sin\gamma\sin\beta$$

reduces the problem to the question when the function

$$\sin\gamma\sin\beta\cos\beta=\tfrac{1}{2}\sin\gamma\sin 2\beta$$

has its extreme values. Obviously

$$\beta_{\min}=0$$

and

$$\beta_{\max}=\gamma$$

This means that we have minimum interference if $\varphi=y_0$ or $\varphi\in R(P_B)$, and maximum interference if $\varphi=x_0$ or $\varphi\in R(P_A)$.

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